Graph Library: Background and Terminology

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1 Getting Started

This paper is one of several interrelated papers for a proposed Graph Library for the Standard C++ Library. The Table 1 describes all the related papers.

Paper	Status	Description
P1709	Inactive	Original proposal, now separated into the following papers.
P3126	Active	**Overview**, describing the big picture of what we are proposing.
P3127	Active	**Background and Terminology** providing the motivation, theoretical background and terminology used across the other documents.
P3128	Active	**Algorithms** covering the initial algorithms as well as the ones we'd like to see in the future.
P3129	Active	**Views** has helpful views for traversing a graph.
P3130	Active	**Graph Container Interface** is the core interface used for uniformly accessing graph data structures by views and algorithms. It is also designed to easily adapt to existing graph data structures.
P3131	Active	**Graph Containers** describing a proposed high-performance `compressed_graph` container. It also discusses how to use containers in the standard library to define a graph, and how to adapt existing graph data structures.

Table 1: Graph Library Papers

Reading them in order will give the best overall picture. If you're limited on time, you can use the following guide to focus on the papers that are most relevant to your needs.

Reading Guide

— If you're **new to the Graph Library**, we recommend starting with the *Overview* paper (P3126) to understand focus and scope of our proposals.

— If you want to **understand the theoretical background** that underpins what we're doing, you should read the *Background and Terminology* paper (P3127).

— If you want to **use the algorithms**, you should read the *Algorithms* paper (P3128) and *Graph Containers* paper (P3131).

— If you want to **write new algorithms**, you should read the *Views* paper (P3129), *Graph Container Interface* paper (P3130) and *Graph Containers* paper (P3131). You'll also want to review existing implementations in the reference library for examples of how to write the algorithms.

— If you want to **use your own graph container**, you should read the *Graph Container Interface* paper (P3130) and *Graph Containers* paper (P3131).

2 Revision History

P3127r0

— Split from the P1709r5 *Overview and Introduction* section and expanded with more details and examples. Also added *Getting Started* section.

3 Naming Conventions

Table 2 shows the naming conventions used throughout the Graph Library documents.

Template Parameter	Type Alias	Variable Names	Description
G			Graph
	graph_reference_t<G>	g	Graph reference
GV		val	Graph Value, value or reference
V	vertex_t<G>		Vertex
	vertex_reference_t<G>	u,v,x,y	Vertex reference. u is the source (or only) vertex. v is the target vertex.
VId	vertex_id_t<G>	uid,vid,seed	Vertex id. uid is the source (or only) vertex id. vid is the target vertex id.
VV	vertex_value_t<G>	val	Vertex Value, value or reference. This can be either the user-defined value on a vertex, or a value returned by a function object (e.g. VVF) that is related to the vertex.
VR	vertex_range_t<G>	ur,vr	Vertex Range
VI	vertex_iterator_t<G>	ui,vi	Vertex Iterator. ui is the source (or only) vertex.
		first,last	vi is the target vertex.
VVF		vvf	Vertex Value Function: vvf(u) → vertex value, or vvf(uid) → vertex value, depending on requirements of the consume algorithm or view.
VProj		vproj	Vertex descriptor projection function: vproj(x) → vertex_descriptor<VId,VV> .
	partition_id_t<G>	pid	Partition id.
		P	Number of partitions.
PVR	partition_vertex_range_t<G>	pur,pvr	Partition vertex range.
E	edge_t<G>		Edge
	edge_reference_t<G>	uv,vw	Edge reference. uv is an edge from vertices u to v . vw is an edge from vertices v to w .
EId	edge_id_t<G>	eid,uvid	Edge id, a pair of vertex_ids.
EV	edge_value_t<G>	val	Edge Value, value or reference. This can be either the user-defined value on an edge, or a value returned by a function object (e.g. EVF) that is related to the edge.
ER	vertex_edge_range_t<G>		Edge Range for edges of a vertex
EI	vertex_edge_iterator_t<G>	uvi,vwi	Edge Iterator for an edge of a vertex. uvi is an iterator for an edge from vertices u to v . vwi is an iterator for an edge from vertices v to w .
EVF		evf	Edge Value Function: evf(uv) → edge value, or evf(eid) → edge value, depending on the requirements of the consuming algorithm or view.
EProj		eproj	Edge descriptor projection function: eproj(x) → edge_descriptor<VId,Sourced,EV> .
PER	partition_edge_range_t<G>		Partition Edge Range for edges of a partition vertex.

Table 2: Naming Conventions for Types and Variables

4 Motivation

The original STL revolutionized the way that C++ programmers could apply algorithms to different kinds of containers, by defining *generic* algorithms, realized via function templates. A hierarchy of *iterators* were the mechanism by which algorithms could be made generic with respect to different kinds of containers, Named requirements specified the valid expressions and associated types that algorithms required of their arguments. As of C++20, we now have both ranges and concepts, which now provide language-based mechanisms for specifying requirements for generic algorithms.

As powerful as the algorithms in the standard library are, the underlying basis for them is a range (or iterator pair), which inherently can only specify a one-dimensional container. Iterator pairs (equiv. ranges) specify a `begin()` and an `end()` and can move between those two limits in various ways, depending on the type of iterator. As a result, important classes of problems that programmers are regularly faced with use structures that are not one-dimensional containers, and so the standard library algorithms can't be directly used. Multi-dimensional arrays are an example of one such kind of data structure. Matrices do have the nice property that they (typically) have the ability to be "raveled", i.e., the data underlying the matrix can still be treated as a one-dimensional container. Multi-dimensional arrays also have the property that, even though they can be thought of as hierarchical containers, the hierarchy is uniform—an N-dimensional array is a container of N-1 dimensional arrays.

Another important problem domain that does not fit into the category of one-dimensional ranges is that of *graph algorithms and data structures*. Graphs are a powerful abstraction for modeling relationships between entities in a given problem domain, irrespective of what the actual entities are, and irrespective of what the actual relationships are. In that sense, graphs are, by there very nature, generic. Graphs are a fundamental abstraction in computer science, and are ubiquitous in real-world applications.

Any problem concerned with connectivity can be modeled as a graph. Just a small set of examples include Internet routing, circuit partitioning and layout, finding the best route to take to a destination on map. There are also relationships between entities that are inferred from large sets of data, for example the graph of consumers who have purchased the same product, or who have viewed the same movie. Yet more interesting structures arise (hypergraphs or k-partite graphs) can arise when we want to model relationships between diverse types of data, such as the graph of consumers, the products they have purchased, and the vendors of the products. And, of course, graphs play a critical role in multiple aspects of machine learning.

On the flip side of graph structures are the graph algorithms that are widely used for problems such as the above. Well-known graph algorithms include breadth-first search, Dijkstra's algorithm, connected components, and so on. Because graphs can come from so many different problem domains, they will also be represented with many different kinds of data structures. To make graph algorithms as usable as possible across arbitrary representation requires application of the same principles that were used in the original STL: a collection of related algorithms from a problem domain (in our case, graphs), minimizing the requirements imposed by the algorithms on their arguments, systematically organizing the requirements, and realizing this framework of requirements in the form of concepts.

There are also many uses of graphs that would not be met by a standard set of algorithms. A standardized interface for graphs is eminently useful in such situations as well. In the most basic case, it would provide a well-defined framework for development. But in keeping with the foundational goal of generic programming to enable reuse, it would also empower users to develop and deploy their own reusable graph components. In the best case, such algorithms would be available to the broader C++ programmer community.

Because graphs are so ubiquitous and so important to modern software systems, a standardized library of graph algorithms and data structures would have enormous benefit to the C++ development community. This proposal contains the specification of such a library, developed using the principles above.

5 Example: Six Degrees of Kevin Bacon

A classic example of the use of a graph algorithm is the game "The Six Degrees of Kevin Bacon." The game is played by connecting actors to each other through movies they have appeared in together. The goal is to find the smallest number of movies that connect a given actor to Kevin Bacon. That number is called the "Bacon

number" of the actor. Kevin Bacon himself has a Bacon number of 0. Since Kevin Bacon appeared with Tom Cruise in "A Few Good Men", Tom Cruise has a Bacon number of 1.

The following program computes the Bacon number for a small selection of actors.

```
std::vector<std::string> actors { "Tom Cruise", "Kevin Bacon", "Hugo Weaving",
                                  "Carrie-Anne Moss", "Natalie Portman", "Jack Nicholson",
                                  "Kelly McGillis", "Harrison Ford", "Sebastian Stan",
                                  "Mila Kunis", "Michelle Pfeiffer", "Keanu Reeves",
                                  "Julia Roberts" };

using G = std::vector<std::vector<int>>;
G costar_adjacency_list{
    {1, 5, 6}, {7, 10, 0, 5, 12}, {4, 3, 11}, {2, 11}, {8, 9, 2, 12}, {0, 1}, {7, 0},
    {6, 1, 10}, {4, 9}, {4, 8}, {7, 1}, {2, 3}, {1, 4} };

int main() {
  std::vector<int> bacon_number(size(actors));

  // 1 -> Kevin Bacon
  for (auto&& [uid,vid] : basic_sourced_edges_bfs(costar_adjacency_list, 1)) {
    bacon_number[vid] = bacon_number[uid] + 1;
  }

  for (int i = 0; i < size(actors); ++i) {
    std::cout << actors[i] << " has Bacon number " << bacon_number[i] << std::endl;
  }
}
```

Output:

```
Tom Cruise has Bacon number 1
Kevin Bacon has Bacon number 0
Hugo Weaving has Bacon number 3
Carrie-Anne Moss has Bacon number 4
Natalie Portman has Bacon number 2
Jack Nicholson has Bacon number 1
Kelly McGillis has Bacon number 2
Harrison Ford has Bacon number 1
Sebastian Stan has Bacon number 3
Mila Kunis has Bacon number 3
Michelle Pfeiffer has Bacon number 1
Keanu Reeves has Bacon number 4
Julia Roberts has Bacon number 1
```

In graph parlance, we are creating a graph where the vertices are actors and the edges are movies. The number of movies that connect an actor to Kevin Bacon is the shortest path in the graph from Kevin Bacon to that actor. In the example above, we compute shortest paths from Kevin Bacon to all other actors and print the results. Note, however, that actor-actor relationships are not how data about actors is available in the wild (from IMDB, for example). Rather, two types of relationships available are actor-movie and movie-actor. See Section 7 below.

6 Graph Background

For clarity, we briefly review some of the basic terminology of graphs. We use commonly accepted terminology for graph data structures and algorithms and adopt the particular terminology used in the textbook by Cormen, Leiserson, Rivest, and Stein ("CLRS") [1].

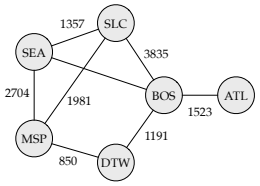
6.1 Basic Terminology

To model the relationships between entities, a *graph* G comprises two sets: a *vertex set* V, whose elements correspond to the entities, and an *edge set* E, whose elements are pairs corresponding to elements in V that have some relationship with each other. That is, if u and v are members of V that have some relationship that we wish to capture, then there is a pair $\{u, v\}$ in E. We can express that together V and E define a graph as $G = \{V, E\}$.

Two examples of graph models are shown in Figures 1a and 1b, which respectively model a network of routes between and an electronic circuit. The figures show the domain-specific data to be modeled and the sets V and E for each graph. Also shown for each graph is a node and link diagram, a commonly-used graphical[1] notation.

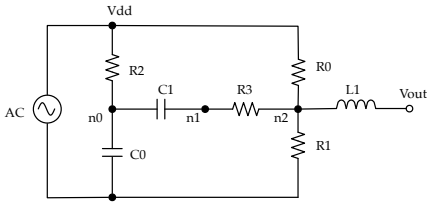
Airport
SEA
MSP
SLC
DTW
ATL
BOS

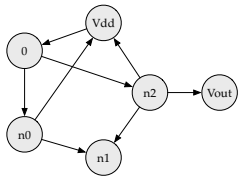
Distance (km)		
MSP	DTW	850
SLC	SEA	1357
MSP	SLC	1981
BOS	SLC	3835
SEA	BOS	4016
BOS	ATL	1523
SEA	MSP	2704
BOS	DTW	1191



$$
\begin{aligned}
G &= \{V, E\} \\
V &= \{\text{SEA}, \text{MSP}, \text{SLC}, \text{DTW}, \text{ATL}, \text{BOS}\} \\
E &= \{\{\text{MSP}, \text{DTW}\}, \{\text{SLC}, \text{SEA}\}, \\
&\quad \{\text{MSP}, \text{SLC}\}, \{\text{BOS}, \text{SLC}\}, \\
&\quad \{\text{SEA}, \text{BOS}\}, \{\text{BOS}, \text{ATL}\}, \\
&\quad \{\text{SEA}, \text{MSP}\}, \{\text{BOS}, \text{DTW}\}\}
\end{aligned}
$$

(a) An undirected graph representing airline routes between cities. Shown are the list of airports (the vertices) and the list of routes between them (the edges). Also shown are a node and link diagram and the set-based description.





$$
\begin{aligned}
G &= \{V, E\} \\
V &= \{0, \text{Vdd}, \text{n0}, \text{n1}, \text{n2}, \text{Vout}\} \\
E &= \{(\text{n0}, \text{n1}), (\text{Vdd}, 0), \\
&\quad (\text{n0}, \text{Vdd}), (\text{n2}, \text{Vdd}), \\
&\quad (0, \text{n2}), (\text{n2}, \text{Vout}), \\
&\quad (0, \text{n0}), (\text{n2}, \text{n1})\}
\end{aligned}
$$

(b) A directed graph representing an electronic circuit. Shown in the circuit diagram are the labeled circuit nodes (the vertices) and the circuit elements connecting the nodes (the edges). Since circuit elements are oriented, we use a directed graph to model the circuit. Also shown are a node and link diagram and the set-based description.

Figure 1: Graph models of an airline route system and of an electronic circuit.

6.2 Graph Representation: Enumerating the Vertices

To reason about graphs, and to write algorithms for them, we require a *representation* of the graph. We note that *a graph and its representation are not the same thing.* It is therefore essential that we be precise about this distinction as we develop a software library of graph algorithms and data structures[2].

The representations that we will be using are familiar ones: adjacency matrix, edge list, and adjacency list. We begin with a process that is so standard that we typically don't even notice it, but it forms the foundation of graph representations: we *enumerate the vertices.* That is, we assign an index to each element of V and write $V = \{v_0, v_1, \ldots v_{n-1}\}$. Based on that enumeration, elements of E are expressed in the form $\{v_i, v_j\}$. Similarly, we can enumerate the edges, and write $E = \{e_0, e_1, \ldots e_{m-1}\}$, though the enumeration of E does not play a role in standard representations of graphs. The number of elements in V is denoted by $|V|$ and the number of elements in E is denoted by $|E|$.

We summarize some remaining terminology about vertices and edges.

— An edge e_k may be *directed*, denoted as the ordered pair $e_k = (v_i, v_j)$, or it may be *undirected*, denoted as the (unordered) set $e_k = \{v_i, v_j\}$. The edges in E are either all directed or all undirected, corresponding respectively to a *directed graph* or to an *undirected* graph.

[1] An unfortunate collision of terminology.

[2] In fact, if we are to be completely precise, the library we are proposing is one of algorithms and data structures for graph representations. We will make concessions to commonly accepted terminology, while precisely defining that terminology.

— If the edge set E of a directed graph contains an edge $e_k = (v_i, v_j)$, then vertex v_j is said to be *adjacent* to vertex v_i. The edge e_k is an *out-edge* of vertex v_i and an in-edge of vertex v_j. Vertex v_i is the *source* of edge e_k, while v_j is the *target* of edge e_k.

— If the edge set E of an undirected graph contains an edge $e_k = \{v_i, v_j\}$, then e_k is said to be *incident* on the vertices v_i and v_j. Moreover, vertex v_j is adjacent to vertex v_i *and* vertex v_i is adjacent to vertex v_j. The edge e_k is an out-edge of both v_i and v_j and it is an in-edge of both v_i and v_j.

— The *neighbors* of a vertex v_i are all the vertices v_j that are adjacent to v_i. The set of all of the neighbors is the *neighborhood* of v_i.

— A *path* as a sequence of vertices $v_0, v_1, \ldots, v_{k-1}$ such that there is an edge from v_0 to v_1, an edge from v_1 to v_2, and so on. That is, a path is a set of edges $(v_i, v_{i+1}) \in E$ for $i = 0, 1, \ldots, k-2$.

6.3 Adjacency-Based Representations

We begin our development of graph representations with the almost universally-accepted definition of the adjacency matrix representation of a graph. The *adjacency matrix representation* of a graph G is a $|V| \times |V|$ matrix $A = (a_{ij})$ such that, respectively for a directed or undirected graph

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases} \qquad a_{ij} = a_{ji} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

That is, $a_{ij} = 1$ if and only if v_j is adjacent to v_i in the original graph G (hence the name "adjacency matrix"). Here we can see why we said that the initial enumeration of V is foundational to representations: *The adjacency matrix is based solely on the indices used in that enumeration.* It does not contain the vertices or edges themselves.

As a data structure to use for algorithms, the adjacency matrix is not very efficient, neither in terms of storage (which, at $|V| \times |V|$ is prohibitive), nor for computation. Instead of storing the entire adjacency matrix, we can simply store the index values of its non-zero elements. A *sparse coordinate adjacency matrix* is a container C of pairs (i, j) for every a_{ij} in A. At first glance, it may seem that we have simply created a data structure C that has a pair (i, j) if E in the original graph has an edge from v_i to v_j. This is true in the directed case. However, in the undirected case, if there is an edge between v_i and v_j, then v_i is adjacent to v_j and v_j is adjacent to v_i. In other words, if there is an edge between v_i and v_j in an undirected graph, then both the entries a_{ij} and a_{ji} are equal to 1[3] — and therefore for a single edge between v_i and v_j, C contains two index pairs: (i, j) and (j, i). The sparse coordinate representation is commonly known as *edge list*. However, we caution the reader that C does not store edges, but rather indices and that, in the case that it represents an undirected graph, there is not a 1-1 correspondence between the edges in E and the contents of C.

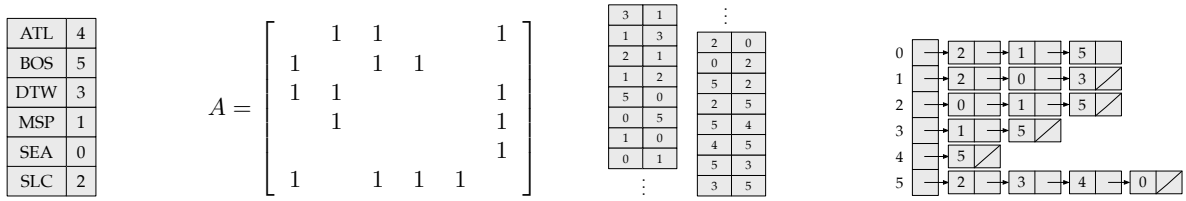
Although the sparse coordinate adjacency matrix is much more efficient in terms of storage than the original adjacency matrix, it isn't as efficient as it could be. Much more importantly, it is not useful for the types of operations used by most graph algorithms, which need to be able to get the set of neighbors of a given vertex in constant time. To support this type of operation, we use a *compressed sparse adjacency matrix*, which is an array J with $|V|$ entries, where each $J[i]$ is a linear container of indices $\{j\}$ such that v_j is a neighbor of v_i in G. That is j is contained in $J[i]$ if and only if there is an edge (v_i, v_j) in E (or, equivalently, if there is a pair (i, j) in C or, equivalently, if $a_{ij} = 1$)[4]. We note that if (v_i, v_j) is an edge in an undirected graph, $J[i]$ will contain j and $J[j]$ will contain i. The common name for this data structure is *adjacency list*. Although this name is problematic (for instance, it is not actually a list), it is so widely used that we also use it here—but *we mean specifically that an "adjacency list" is the compressed sparse adjacency matrix representation of a graph*[5]. Again we emphasize the distinction between a graph and its representation: An adjacency list J is not the same as the graph G—it is a representation of G.

Illustrations of the adjacency-matrix representations of the airline route graph and the electronic circuit graph are shown in Figures 2 and 3, respectively.

[3] That is, the adjacency matrix is symmetric.

[4] The compressed sparse adjacency matrix is identical to the compressed sparse row format from linear algebra

[5] We concede that "adjacency list" rolls off the tongue much more easily than "compressed sparse adjacency matrix representation of a graph."



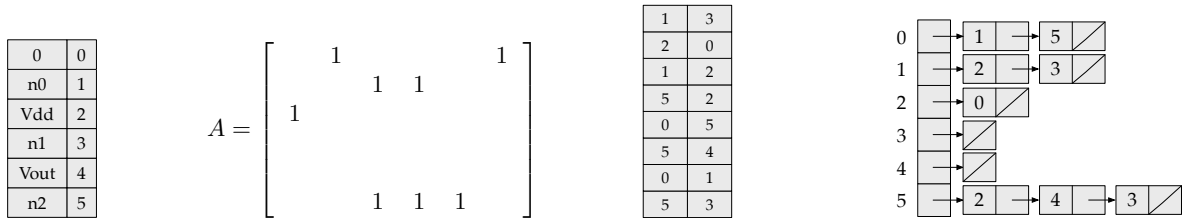
(a) An enumeration of the airport graph given in 1a.

(b) The adjacency matrix representation of the graph given in Figure 1a, using the enumeration given in Figure 2a.

(c) The coordinate sparse adjacency matrix representation (shown split into two columns).

(d) The compressed sparse adjacency matrix representation.

Figure 2: Adjacency matrix representations of the airport graph model.



(a) An enumeration of the circuit graph given in 1b.

(b) The adjacency matrix representation of the graph given in Figure 1b, using the enumeration given in Figure 3a.

(c) The coordinate sparse adjacency matrix representation.

(d) The compressed sparse adjacency matrix representation.

Figure 3: Adjacency matrix representations of the circuit graph model.

7 Bipartite Graphs

So far, we have been considering graphs where edges in E are pairs of vertices, which are taken from a single set V. We refer to such a graph as a *unipartite* graph. But consider again the Kevin Bacon example. The source for the information comprising the Kevin Bacon data is the Internet Movie Database (IMDB). However, the IMDB does not contain any explicit information about the relationships between actors. Rather it contains files of tabular data, one of which contains an entry for each movie with the list of actors that have appeared in that movie, and another of which contains an entry for each actor with the list of movies that actor has appeared in ("movie-actor" and "actor-movie" tables, respectively). Such tables are shown in Figure 4.[6] Thus, a graph, as we have defined it, cannot model the IMDB.

There is a small generalization we can make to the definition of graph that will result in a suitable abstraction for modeling the IMDB. In particular, we need one set of vertices corresponding to actors, another set of vertices corresponding to movies, and then a set of edges corresponding to the relationships between actors and movies. There are two kinds of relationships to consider actors in movies or movies starring actors. To be well-defined, the edge set may only contain one kind of relationship. To capture this kind of model, we define a *structurally bipartite graph* $H = \{U, V, E\}$, where vertex sets U and V are enumerated $U = \{u_0, u_1, \ldots, u_{n0}\}$ and $V = \{v_0, v_1, \ldots v_{n1}\}$, and the edge set E consists of pairs (u_i, v_j) where u_i is in U and v_j is in V.

The *adjacency matrix representation of a structurally bipartite graph* is a $|U| \times |V|$ matrix $A = (a_{ij})$ such that,

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

From this adjacency matrix representation we can readily construct coordinate and compressed sparse representations. The only structural difference between the representations of a structurally bipartite graph and that

[6] This is a greatly simplified version of the CSV files that actually comprise the IMDB. The full set of files is available for non-commercial use at https://datasets.imdbws.com.

0	Tom Cruise
1	Kevin Bacon
3	Carrie-Ann Moss
4	Natalie Portman
2	Hugo Weaving
5	Kelly McGillis

(a) Table of actors.

0	A Few Good Men
1	Top Gun
2	V for Vendetta
3	Black Swan
4	The Matrix

(b) Table of movies.

Tom Cruise	Top Gun, A Few Good Men
Kevin Bacon	A Few Good Men
Hugo Weaving	V for Vendetta, The Matrix
Carrie-Ann Moss	The Matrix
Natalie Portman	Black Swan, V for Vendetta
Kelly McGillis	Top Gun

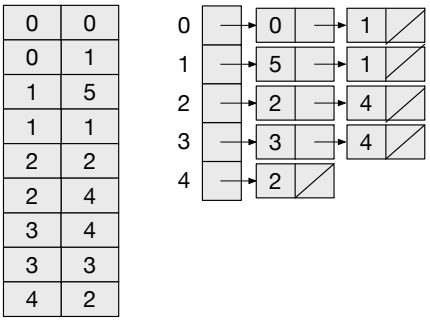
(c) A table of actors and movies they have appeared in.

A Few Good Men	Tom Cruise, Kevin Bacon
Top Gun	Kelly McGillis, Tom Cruise
V for Vendetta	Hugo Weaving, Natalie Portman
Black Swan	Natalie Portman
The Matrix	Carrie-Ann Moss, Hugo Weaving

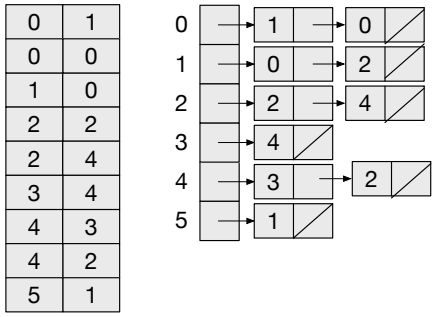
(d) A table of movies with starring actors.

Figure 4: Illustrative simplification of IMDB actor and movie data.

of a unipartite graph is that of vertex cardinality. That is, in a unipartite graph, edges map from V to V, and hence the values in the left hand column and in the right hand column of a coordinate representation would be in the same range: $[0, |V|)$. However, for a structurally bipartite graph, this is no longer the case. Although the coordinate representation still consists of pairs of vertex indices, the range of values in the left hand column is $[0, |U|)$, while in the right hand column it is $[0, |V|)$. Similarly, the compressed representation will have $|U|$ entries, but the values stored in each entry may range from $[0, |V|)$. We note that these are constraints on values, not on structure.



(a) Coordinate and compressed sparse adjacency representations for movies with their starring actors.



(b) Coordinate and compressed sparse adjacency representations for actors and the movies they have appeared in.

Figure 5: Sparse adjacency representations (edge lists and adjacency lists) for IMDB actor and movie data.

We distinguish a structurally bipartite graph from simply a bipartite graph because the former applies separate enumerations to U and V. In customary graph terminology, a *bipartite* graph is one in which the vertices can be partitioned into two disjoint sets, such that all of the edges in the graph only connect vertices from one set to vertices of the other set. However, although the vertices are partitioned, they are still taken from the same original vertex set V and have a single enumeration. Whether a graph can be partitioned in this way is a run-time property inherent to the graph itself (which can be discovered with an appropriate algorithm). This is not a natural way to model separate categories of entities, such as movies and actors, where entities are categorized completely independently of each other and it is therefore most appropriate to have independent enumerations for them. A structurally bipartite graph explicitly captures distinct vertex categories.

8 Partitioned Graphs

In contrast to structurally bipartite graphs, there are certainly cases where one would want to maintain two categories of entities, or otherwise distinguish the vertices, from the same vertex set. In that case, we would use a *partitioned graph*, which we define as $G = \{V, E\}$, where the vertex set V consists of non-overlapping subsets, i.e., $V = \{V_0, V_1, \ldots\}$ which we enumerate as $V_0 = \{v_0, v_1, \ldots, v_{n0-1}\}$, $V_1 = \{v_{n0}, \ldots, v_{n1-1}\}$ and so on. Each V_i is a *partition* of V. The total enumeration of V is $V = \{v_0, v_1, \ldots, v_{n-1}\}$. Just as each V_i is a partition of V, the enumeration of each V_i is a partitioning of the enumeration of V.

The edge set E still consists of edges (v_i, v_j) (or $\{v_i, v_j\}$ where, in general, v_i and v_j may come from any partition.

We note that partitioned graphs are not restricted to two partitions—a partitioned graph can represent an arbitrary number of partitions, i.e., a *multipartite* graph (a graph with multiple subsets of vertices such that edges only go between subsets). While partitioned graphs can be used to model multipartite graphs, partitioned graphs are not necessarily multipartite; edges can comprise vertices within a partition as well as well as across partitions.

9 From Data to Graph

9.1 Columnar Data

Here we show how one might create an unlabeled edge list from a table of data stored in a CSV file. The following loads a list of directed edges from a CSV file (the values in each row are assumed to be separated by whitespace)[7]. The elements of the first column are considered to be the source vertices and the elements of the second column are the destination vertices. If the edges also had properties, the third column would contain the property values. In this example, the edges are loaded into a vector of tuples, which meets the requirements of a (presumed) `sparse_coordinate` concept.

```
auto sparse_coordinate edges = std::vector<std::tuple<vertex_id_t, vertex_id_t>;
auto input = std::ifstream ("input.csv");
vertex_id_t src, dst;
while (input >> src >> dst) {
    edges.emplace_back (src, dst);
}
```

Similarly, we could load a list of undirected edges from a CSV file into a `sparse_coordinate` structure. Note that, as discussed above, the coordinate sparse adjacency matrix representation (aka an edge list), contains an entry (i, j) as well as an entry (j, i) for each undirected edge $\{v_i, v_j\}$. Hence, we add both `(src, dst)` and `(dst, src)` to `edges`.

```
auto sparse_coordinate edges = std::vector<std::tuple<vertex_id_t, vertex_id_t, double> edges;
auto input = std::ifstream ("input.csv");
vertex_id_t src, dst;
double val;
while (input >> src >> dst >> val) {
    edges.emplace_back (src, dst, val);
    edges.emplace_back (dst, src, val);
}
```

These examples are meant to be illustrative and not necessarily comprehensive (nor efficient). There are, of course, many ways to define containers that meet the requirements of the edge list concept and many ways to create an edge list from columnar data.

9.2 Converting an Edge List to an Adjacency List

The following creates a compressed sparse representation (an adjacency list) from a coordinate sparse representation. The adjacency list is represented as a `std::vector<std::vector<vertex_id_t>>`;

[7] We take a broad view of what a comma is.

```
auto sparse_coordinate edges = std::vector<std::tuple<vertex_id_t, vertex_id_t>;
// Read the edges
auto sparse_compressed adj_list = std::vector<std::vector<vertex_id_t>>;
for (auto [src, dst] : edges) {
  if (src >= adj_list.size()) {
    adj_list.resize(src + 1);
  }
  adj_list[src].push_back (dst);
}
```

We note that the `sparse_coordinate` representation is agnostic as to whether it was originally created based on directed edges or undirected edges. An optimization to the sparse coordinate representation would be to use a *packed coordinate* representation, which would only maintain a single entry for each undirected edge. In that case, we would need to have two complementary insertions into the adjacency list for each entry in the packed coordinate representation.

The following example illustrates the use of a packed coordinate format to construct an adjacency list with an edge property.

```
auto packed_sparse_coordinate edges = std::vector<std::tuple<vertex_id_t, vertex_id_t, double>>;
// Read the edges
auto compressed_sparse adj_list = std::vector<std::vector<std::tuple<vertex_id_t, double>>>(edges
    .num_vertices();
for (auto [src, dst, val] : edges) {
  adj_list[src].push_back (dst, val);
  adj_list[dst].push_back (src, val);
}
```

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